ON THE DERIVATION OF THE EQUATIONS OF ANISOTROPIC MAGNETOHYDRODYNAMICS

(K VYVODU URAVNENII ANIZOTROPNOI MAGNITNOI GIDRODINAMIKI)

PMM Vol.26, No.6, 1962, pp. 1092-1093

V. B. BARANOV
(Moscow)

(Received September 13, 1962)

In the case of strong magnetic fields, the viscous stress tensor and the heat flow vector in magnetohydrodynamics depend on the magnetic field.

A two-fluid study of a fully ionized gas was carried out in [1], where the viscous stress tensors and the heat flow vectors of the ions and the electrons are calculated separately. Moreover, it is assumed that the temperature T_1 of the electron gas differs from T_2 of the ions.

In [2], based on the results of [1], expressions are obtained for the components of the viscous stress tensor and the heat flow vector in the one-fluid approximation. It is also assumed that

$$T_1 = T_2 = T \text{ and } \sqrt{m_1} \ll \sqrt{m_2} \quad (m_1 \text{ and } m_2 \text{ are the masses of the electrons and ions})$$

However, [2] contains certain errors, which, if uncorrected, make the resulting equations inapplicable. In particular, it is necessary to recompute the transfer coefficients in the expressions for the components of the heat-flow vector. Furthermore, corrections were made of some misprints contained in [1], for which the author is indebted to a communication from S.I. Braginskii.

In expression (3.18) for $\mathbf{q}_u$ in [1], instead of the minus sign in front of the curly brackets, there is a plus sign. In formulas (4.14), instead of b'', read $-b''$. In [2], in describing the viscous stress tensor, errors made by Chapman and Cowling [3] and corrected in [4], were repeated in the signs of certain terms.

Using the same notation as in [2], we write the expressions for the components of the viscous stress tensor in the one-fluid approximation (furthermore, we consider only fully ionized gases consisting of electrons and monatomic ions)

$$
\begin{aligned}
\pi_{zz} &= -\eta e_{zz} \\
\pi_{xx} &= -\eta\left\{b'_2 e_{xx} + \frac{1}{2}(b'_2 - 1)\, e_{zz} + 2\omega_2\tau_2 b''_2 e_{xy}\right\} \\
\pi_{yy} &= -\eta\left\{b'_2 e_{yy} + \frac{1}{2}(b'_2 - 1)\, e_{zz} - 2\omega_2\tau_2 b''_2 e_{xy}\right\} \qquad (1) \\
\pi_{xy} &= \pi_{yx} = -\eta\,\{b'_2 e_{xy} - b''_2\omega_2\tau_2\,(e_{xx} - e_{yy})\} \\
\pi_{xz} &= \pi_{zx} = -\eta\,\{b'_1\, e_{xz} + b''_1\omega_2\tau_2 e_{yz}\} \\
\pi_{yz} &= \pi_{zy} = -\eta\{b'_1\, e_{yz} - b''_1\omega_2\tau_2 e_{xz}\}
\end{aligned}
$$

The values of the coefficients here are the same as in [2]. To obtain the expressions for the components of the heat flow vector, we use the formula in [2]

$$
q_i = \sum_s q_{si} + \frac{5}{2}\sum_s n_s kT c_{si} + \sum_{s,\,k} \pi_{sik} c_{sk} + \frac{1}{2}\sum_s n_s m_s c_{si} c^2_s
$$

$$
\mathbf{c}_s = \mathbf{v}_s - \mathbf{v} \qquad (s = 1, 2)
$$

Since $\mathbf{c}_s$ is proportional to the current density, then, neglecting the last term as a quadratic term in $\mathbf{j}$, we also neglect the third term on the right by comparing it with the second. This is permissible for a continuous medium, when the characteristic frequency Ω is much smaller than the "collision frequency" of the charged particles, if, using some estimates of [5], we assume that $(\Omega/\omega_2)\tau_1/t$ is not greater than unity in order of magnitude. (t is the characteristic time of the problem.) Moreover, it is assumed that $\mathbf{v} \approx \mathbf{v}_2$ (this being true for $m_1 v_1 \ll m_2 v_2$).

Utilizing these corrected results of [1] we may obtain for the components of the heat flow vector in the one-fluid approximation, to first order accuracy in the current density

$$
q_z = -\lambda\left(\frac{\partial T}{\partial z} + \iota j_z\right), \qquad
\begin{aligned}
q_x &= -\lambda\left(\varkappa\frac{\partial T}{\partial x} - \omega_1\tau_1\varkappa'\frac{\partial T}{\partial y} + \iota' j_x - \omega_1\tau_1\iota'' j_y\right) \\
q_y &= -\lambda\left(\varkappa\frac{\partial T}{\partial y} + \omega_1\tau_1\varkappa'\frac{\partial T}{\partial x} + \iota' j_y + \omega_1\tau_1\iota'' j_x\right)
\end{aligned} \qquad (2)
$$

Here

$$
\begin{aligned}
&\lambda = 1.58\,\frac{pk\tau_1}{m_1}, \qquad \varkappa = (1.47\omega_1^2\tau_1^2 + 3.77)\,\Delta_1 + \frac{\omega_2\tau_2}{\omega_1\tau_1}(0.633\,\omega_2^2\tau_2^2 + 0.837)\,\Delta_2 \\
&\omega_1 = \frac{eH}{m_1 c} > 0\,, \qquad \varkappa' = (0.791\omega_1^2\tau_1^2 + 6.86)\,\Delta_1 - \left(\frac{\omega_2\tau_2}{\omega_1\tau_1}\right)^2 (0.791\omega_2^2\tau_2^2 + 1.47)\,\Delta_2 \qquad (3) \\
&\iota = 2.03\,\frac{TH}{\omega_1\tau_1 pc}\,, \qquad \iota' = (1.58\omega_1^4\tau_1^4 + 26.6\omega_1^2\tau_1^2 + 7.66)\,\frac{TH\Delta_1}{\omega_1\tau_1 pc}
\end{aligned}
$$

$$\iota'' = (0.949\omega_1^2\tau_1^2 + 1.93)\,\frac{TH\Delta_1}{\omega_1\tau_1 pc} \qquad (\,|\,e_1\,| = e)$$

$$\frac{1}{\Delta_1} = (\omega_1^4\tau_1^4 + 14.79\omega_1^2\tau_1^2 + 3.77), \qquad \frac{1}{\Delta_2} = (\omega_2^4\tau_2^4 + 2.70\omega_2^2\tau_2^2 + 0.677)$$

We observe that, aside from correcting some signs and values of the coefficients, the nature of the dependence of ι' on $\omega_1\tau_1$ obtained here is different from that in [2], and for large values of $\omega_1\tau_1$, it agrees with the results of [4] to within the coefficients. We recall that the magnetic field H is directed along the z-axis.

Substituting (1) and (2) into the energy equation

$$\rho c_v \frac{dT}{dt} = -p\,\mathrm{div}\,\mathbf{v} - \pi_{\alpha\beta}\nabla_\alpha v_\beta - \nabla\mathbf{q} + \mathbf{j}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right)$$

we finally obtain

$$\rho c_v \frac{dT}{dt} = -p\,\mathrm{div}\,\mathbf{v} + \mathbf{j}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right) + \eta b_2'\left[2\sum_i\left(\frac{\partial v_i}{\partial x_i}\right)^2 - (\mathrm{div}\,\mathbf{v})^2\right] +$$
$$+ \eta b_2'\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2 + \eta b_1'\left[\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)\right] + \frac{1}{3}\eta(\mathrm{div}\,\mathbf{v})^2 +$$
$$+ \eta(b_2' - 1)\frac{\partial v_z}{\partial z}\left(2\,\mathrm{div}\,\mathbf{v} - 3\frac{\partial v_z}{\partial z}\right) + \mathrm{div}(\lambda\varkappa\nabla T) + \frac{\partial}{\partial z}\lambda(1 - \varkappa)\frac{\partial T}{\partial z} + \mathrm{div}(\lambda\iota'\mathbf{j}) + \qquad (4)$$
$$+ \frac{\partial}{\partial z}\lambda(\iota - \iota')j_z - \frac{\partial}{\partial x}\left(\lambda\omega_1\tau_1\varkappa'\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\lambda\omega_1\tau_1\varkappa'\frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial x}(\lambda\omega_1\tau_1\iota'' j_y) + \frac{\partial}{\partial y}(\lambda\omega_1\tau_1\iota'' j_x)$$

If the currents are absent, and the magnetic field $\mathbf{H} = 0$, then equation (4) becomes the energy equation of ordinary hydrodynamics.

In addition to comparing with ordinary heat conduction when $\mathbf{H} = \mathbf{j} = 0$, the terms in (2) describe well-known physical phenomena (e.g. [6]): the terms with ι and ι' give Thomson's effect, those with $\omega_1\tau_1\kappa'$ the Leduc-Rigi effect, and those with $\omega_1\tau_1\iota''$ the Ettingshausen effect. The last two effects are connected with the Larmor rotation of the electrons in the magnetic field.

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Translated by C.K.C.